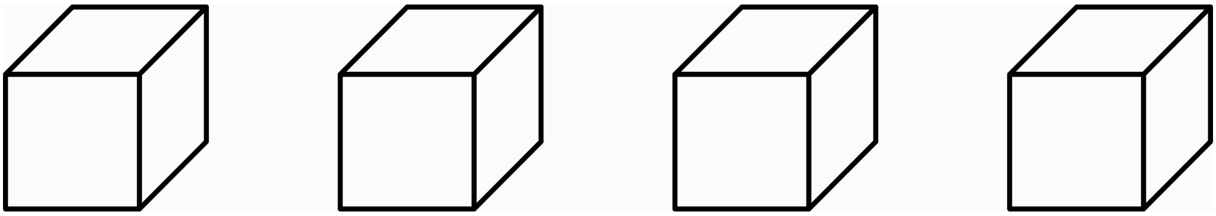
Sum of cubic numbers, calculated in 4 dimensions

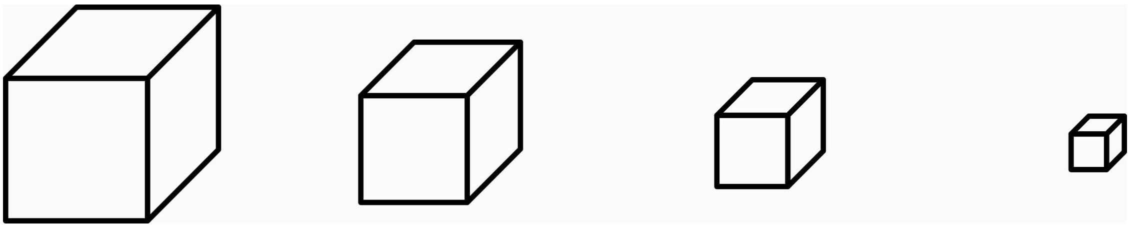
(Felix Voigt, 04.08.23)

While having lunch during rain in the car, I had an idea how to represent the 4th dimension. (Probably this is already well known or I have only remembered it from some incident in the past.) As the things you want to draw are normally restricted to a limited range, it is possible to use the x-axis a second time to represent the 4th dimension. If the 4-dimensional object is limited to the range [0, 1] in dimension 4, than one just draws slices of the object at values 0, 0.25, 0.50, 0.75. That is sufficient, if the boundaries of the object vary in a linear way.

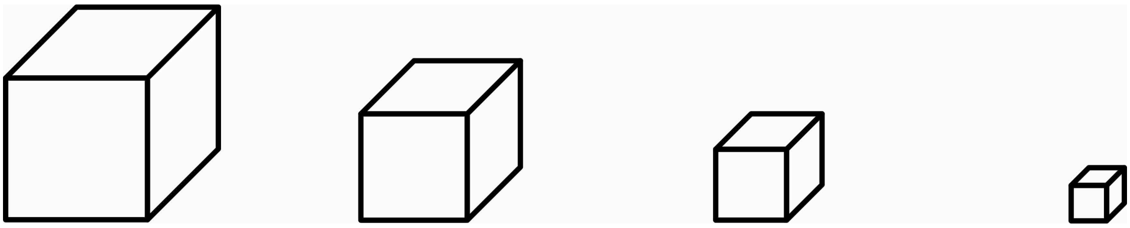
A 4-dimensional cube would look the following way:



A 4-dimensional symmetric pyramid would look like this:

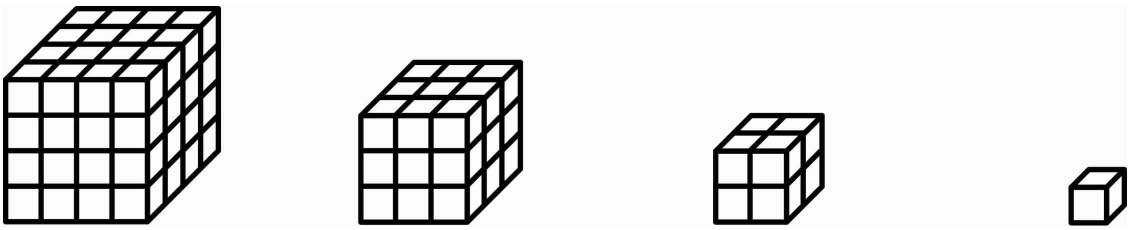


And a special case of an oblique 4d-pyramid could look like this:

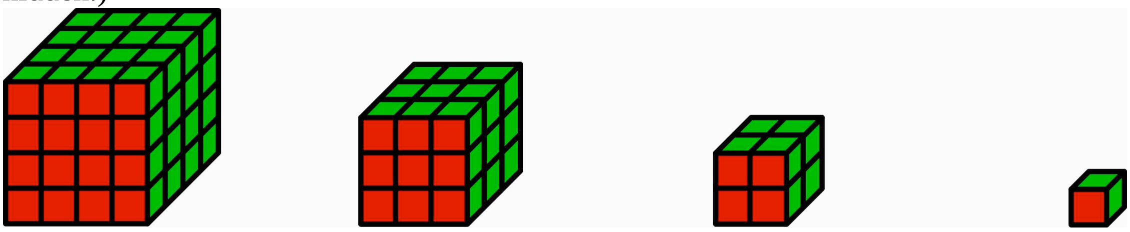


This is an oblique pyramid, where the upper corner (in 4th dimension) is defined as $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

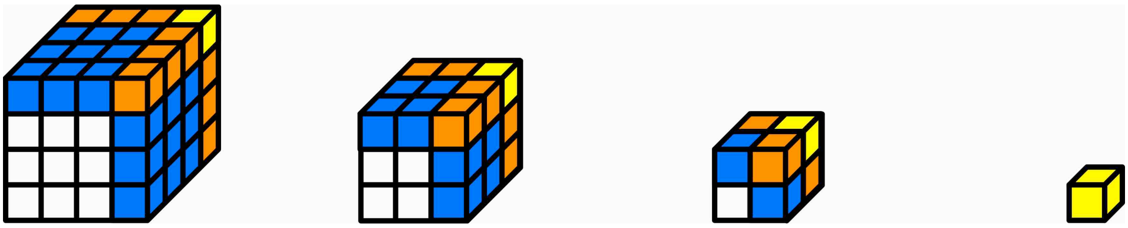
The 4d "quasi-pyramid" which can be used to calculate the sum $1^3+2^3+3^3+4^3$ looks this way:



Analogously to the 3d-pyramid having 4 side-faces, which correspond to the 4 sides of a square, the 4d-pyramid has 6 side-volumes corresponding to the 6 faces of a cube. The little 4d-cubes inside the 4d-quasipyramid are cut only by those side-volumes of the 4d-pyramid, which do not include the point $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$. There are 3 side-volumes, which include this point and 3 side-volume which don‘t. The ones which include the point are colored in red in the following figure and the ones which do not in green. (The side-volumes extend in 4 dimensions, thus e.g. all four green-colored upper sides of the apparent four cubes belong to the same side-volume. Two red-colored side-volumes are not visible in the figure, alsoone green-colored is hidden.)



Now, we color the little cubes according to wether they are attached to one (blue), two (orange) or three (yellow) of the “green” side-volumes.



The little 4d-cubes, corresponding to the blue-colored cubes are cut by a side-volume of the real pyramid into 2 equal parts. The orange-colored unit-4d-cubes are cut by 2 side-volumes such that 1/3 of a unit cube remains inside the real 4d-pyramid. The yellow-colored cubes are cut by 3 side-volumes 3 times, such that inside the real pyramid only ¼ of a unit cube remains, resembling the whole 4d-pyramid.

For the case $n=4$ the sum $S_4=1^3+2^3+3^3+4^3$ can by approximated by $V_{pyr}=\frac{1}{4}base\cdot height=\frac{1}{4}4^3\cdot 4=64$. In order to get the exact result, one has to add the missing pieces. These are

$$V_{blu}=3\cdot(3^2+2^2+1^2)\cdot\frac{1}{2}=3\cdot 14\cdot\frac{1}{2}=21\ ,$$

$$V_{orange}=3\cdot(3+2+1)\cdot\frac{2}{3}=3\cdot 6\cdot\frac{2}{3}=12\ ,$$

$$V_{yello}=1\cdot(1+1+1+1)\cdot\frac{3}{4}=\frac{9}{4}=3\ ,$$

In total one obtains
$S_4=64+21+12+3=100$.

This is the correct result, as
$S_4=1^3+2^3+3^3+4^3=1+8+27+64=100$.

Now we generalize this to the case of n summands.

$$V_{pyr}=\frac{1}{4}base\cdot height=\frac{1}{4}n^3\cdot n=\frac{1}{4}n^4\ ,$$

$$\begin{aligned}V_{blu}&=3\cdot\left(1^2+2^2+3^2+...+[n-1]^2\right)\cdot\frac{1}{2}=\frac{3}{2}\cdot\left(\frac{1}{3}\left([n-1]^3\right)+\frac{1}{2}[n-1]^2+\frac{1}{6}[n-1]\right)\\&=\frac{1}{2}[n^3-3n^2+3n-1]+\frac{3}{4}[n^2-2n+1]+\frac{1}{4}[n-1]\\&=\frac{1}{2}n^3-\frac{3}{4}n^2+\frac{1}{4}n\end{aligned}\quad ,$$

$$V_{orange}=3\cdot\left(1+2+3+...+[n-1]\right)\cdot\frac{2}{3}=2\cdot\frac{[n-1]\cdot n}{2}=n^2-n\ ,$$

$$V_{yello}=1\times\left(1+1+1+...+1\right)\{\text{n times}\}\cdot\frac{3}{4}=\frac{3}{4}n\ .$$

They sum up to

$$S_n=\frac{1}{4}n^4+\left(\frac{1}{2}n^3-\frac{3}{4}n^2+\frac{1}{4}n\right)+\left(n^2-n\right)+\frac{3}{4}n=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2\ \text{, q.e.d.}$$

The result can be checked by a book of formulae, e.g. ″Taschenbuch der Mathematik für Ingenieure und Studenten der Technischen Hochschulen“ von I. N. Bronstein und K. A. Semendjajew, Teubnersche Verlagsgesellschaft, vierte Auflage (1961), S. 137, where it is given in factorized form.

$$S_n=\frac{1}{4}n^2\cdot(n+1)^2=\frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2\ .$$