Magnetic field within the plane of current ring

There are many calculations of the magnetic field on the axis of a current loop on symmetry axis of the loop. There a few calculations of the field of such a loop within the plane of the loop[1,2]. And so far I found no publication on the internet, where the magnetic field is calculated via the vector potential. This is, why I write down and post my derivation on the net.

First, I performed these calculations, because I thought that the derivation via the vector potential would be simpler than the calculation, where the magnetic field is calculated directly via Biot-Savart’s law. Second, I found in the end that in this case, you are prone to commit an error – which I struggled about for some days – and in the end, if one performs the calculations correctly, they will be more complicated. You will see …

The vector potential is defined by the way the magnetic flux density is calculated from it:

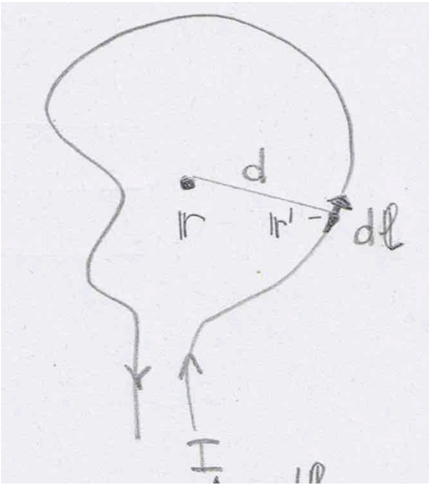
$$\mathbb{B} = \mathrm{rot}\, \mathbb{A}$$

There exists a law to calculate the vector potential from a current density distribution similar to Biot-Savart’s law:

$$\mathbb{A}(\mathbb{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \, \frac{\mathbb{j}(\mathbb{r}')}{|\mathbb{r} - \mathbb{r}'|}$$

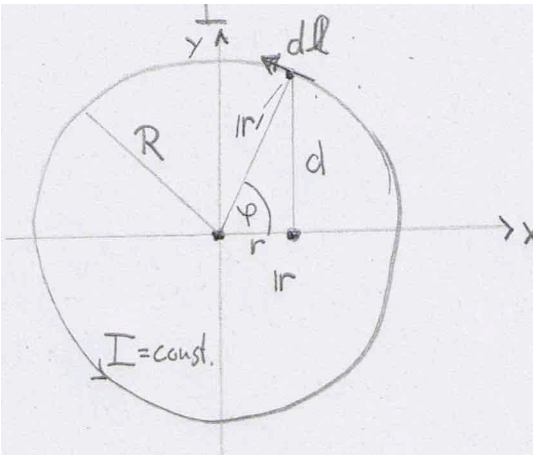
$$= \frac{\mu_0}{4\pi} \int \frac{I \cdot d\ell}{d}$$

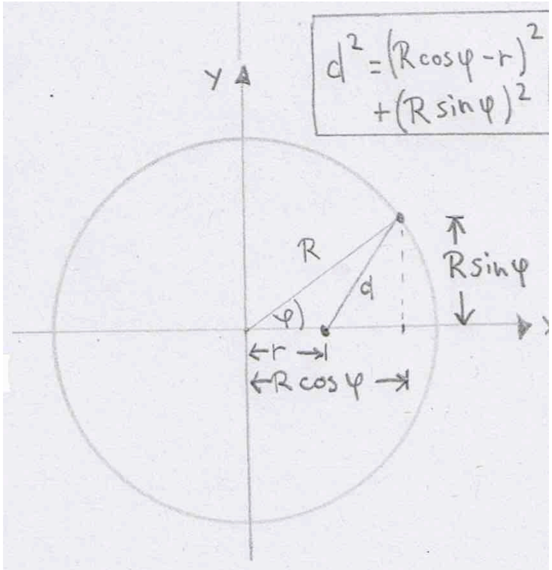
Here, the current density times the volume element has been displaced by the current through the wire times a vectorized length element, which is illustrated in the picture below.



1 https://www.usna.edu/Users/physics/mungan/_files/documents/Scholarship/CurrentLoop.pdf
2 https://phys.libretexts.org/Bookshelves/Electricity_and_Magnetism/
Bo. .._the_Axis_and_in_the_Plane_of_a_Plane_Circular_Current-carrying_Coil

Now, we will calculate the vector potential inside the plane of a current loop at a point sitting at place x=r on the x-axis (see following to pictures).





The three components of the vector potential are:

$$A_x(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{I\, d\ell_x}{d} = 0,$$

$$\text{da } d\ell_x(-\varphi) = -d\ell_x(\varphi) \wedge d(-\varphi) = d(\varphi).$$

$$A_y(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{I\, d\ell_y}{d}$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\varphi \frac{R\cos\varphi}{\{(R\cos\varphi - r)^2 + (R\sin\varphi)^2\}^{1/2}}$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{2\pi} d\varphi \frac{R\cos\varphi}{\{R^2 - 2rR\cos\varphi + r^2\}^{1/2}}$$

$$= \frac{\mu_0 I R}{2\pi}\int_0^{\pi} d\varphi \frac{\cos\varphi}{\{R^2 - 2rR\cos\varphi + r^2\}^{1/2}}$$

$$A_z(\mathbf{r}) = 0$$

Now, we can calculate the magnetic flux density, right?

$$\mathbb{B} = \mathrm{rot}\,\mathbb{A} = \nabla \times \mathbb{A} = \begin{vmatrix} e_x & \partial_x & A_x \\ e_y & \partial_y & A_y \\ e_z & \partial_z & A_z \end{vmatrix}$$

$$= \begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix}$$

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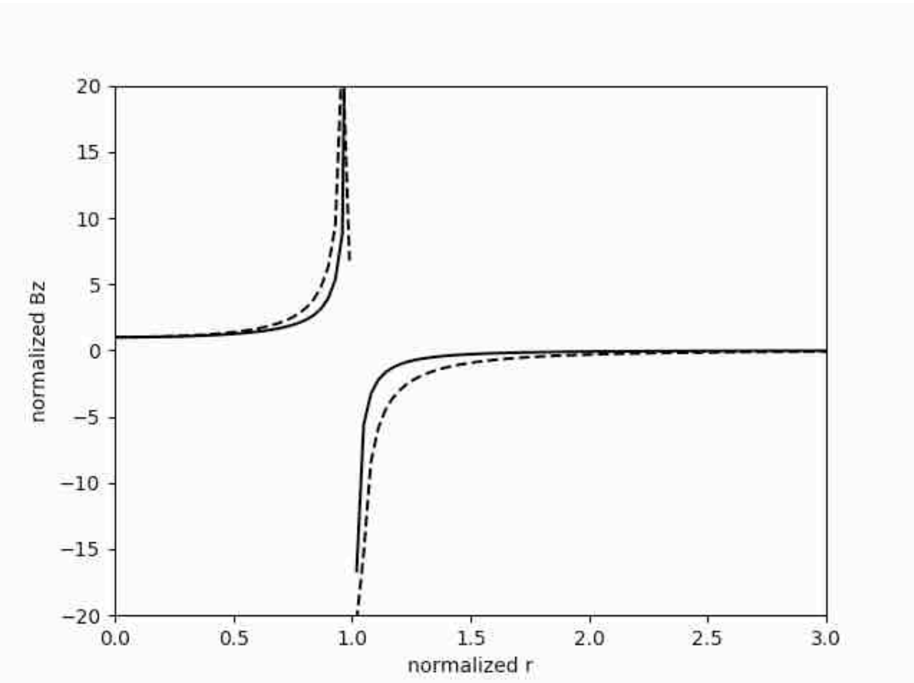
$$\mathbb{B} = \begin{pmatrix} -\partial_z A_y \\ 0 \\ \partial_x A_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \partial_x A_y \end{pmatrix}$$

$$B_z = \frac{\partial A_y}{\partial x}$$

$$B_z(r) = \frac{\mu_0 I R}{2\pi} \frac{\partial}{\partial r} \int_0^{\pi} d\varphi \, \frac{\cos(\varphi)}{\{R^2 - 2rR\cos\varphi + r^2\}^{1/2}}$$

But, this result is not correct. The numerically calculated graph of the normalized B-field vs. radius r (dashed line) is similar as the result shown in refs. 1 (solid line), but it deviates slightly.



Reason for this is, that although Ax(r) is zero along the x-axis, it is not zero in the whole xy-plane. So, the derivative of Ax with respect to y will not be zero and cannot be left out in the calculation of Bz. Even not on the x-axis, where Ax vanishes.

Do we have to do it all over again? No, because of symmetry, one can find the vector potential in xy-plane starting from the values of the vector potential on the x-axis. But, I make a mistake, when performing the calculations following cylindrical symmetry and so far I have not found the reason for the error.

So we do it all over again and calculate the vector potential in the total xy-plane:

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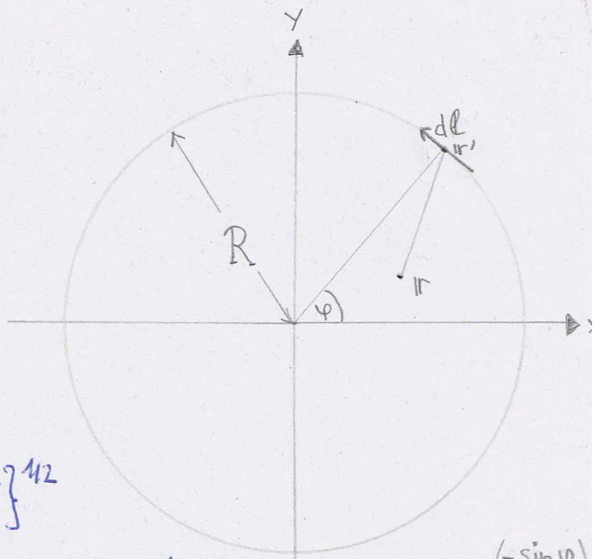
$$\mathbf{r} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \qquad \mathbf{r}' = R\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}$$



$$|\mathbf{r} - \mathbf{r}'| = \left|\begin{pmatrix} x - R\cos\varphi \\ y - R\sin\varphi \\ 0 \end{pmatrix}\right|$$

$$= \left\{(x - R\cos\varphi)^2 + (y - R\sin\varphi)^2\right\}^{1/2}$$

$$= \left\{x^2 + y^2 + R^2 - 2xR\cos\varphi - 2yR\sin\varphi\right\}^{1/2}$$

$$d\boldsymbol{\ell} = R\,d\varphi \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$\frac{A_x(\mathbf{r})}{\mu_0/4\pi} = \int d^3r' \frac{j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$= \int \frac{I\,d\ell_x}{\left\{x^2 + y^2 + R^2 - 2xR\cos\varphi - 2yR\sin\varphi\right\}^{1/2}}$$

$$= \int_0^{2\pi} \frac{-IR\sin\varphi\,d\varphi}{\left\{x^2 + y^2 + R^2 - 2xR\cos\varphi - 2yR\sin\varphi\right\}^{1/2}}$$

$$\frac{A_y(\mathbf{r})}{\mu_0/4\pi} = \ldots$$

$$= \int_0^{2\pi} \frac{IR\cos\varphi\,d\varphi}{\left\{x^2 + y^2 + R^2 - 2xR\cos\varphi - 2yR\sin\varphi\right\}^{1/2}}$$

$$A_z(\mathbf{r}) = 0$$

From the values of Ax and Ay we can calculate the magnetic flux density B. B has only a component in z-direction and, because of cylindrical symmetry, we set y = 0 in the end:

$$\mathbb{B} = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{pmatrix}$$

$$\frac{4\pi}{\mu_0 I R} \cdot \frac{\partial A_y}{\partial x}$$

$$= \frac{\partial}{\partial x} \int_0^{2\pi} d\varphi \cos\varphi \cdot \left\{x^2 + y^2 + R^2 - 2xR\cos\varphi - 2yR\sin\varphi\right\}^{-1/2}$$

$$= \int_0^{2\pi} d\varphi \cos\varphi \left(-\tfrac{1}{2}\right) \left\{x^2 + y^2 + R^2 - 2xR\cos\varphi - 2yR\sin\varphi\right\}^{-\frac{3}{2}} \cdot (2x - 2R\cos\varphi)$$

$$= \Big|_{y=0} \int_0^{2\pi} d\varphi \frac{R\cos^2\varphi - x\cos\varphi}{\left\{x^2 + R^2 - 2xR\cos\varphi\right\}^{3/2}}$$

$$-\frac{4\pi}{\mu_0 I R} \cdot \frac{\partial A_x}{\partial y}$$

$$= \frac{-\partial}{\partial y} \int_0^{2\pi} d\varphi (-\sin\varphi) \left\{x^2 + y^2 + R^2 - 2xR\cos\varphi - 2yR\sin\varphi\right\}^{-1/2}$$

$$= \int_0^{2\pi} d\varphi \sin\varphi \left(-\tfrac{1}{2}\right) \left\{x^2 + y^2 + R^2 - 2xR\cos\varphi - 2yR\sin\varphi\right\}^{-\frac{3}{2}} \cdot (2y - 2R\sin\varphi)$$

$$= \Big|_{y=0} \int_0^{2\pi} d\varphi \frac{R\sin^2\varphi - 0}{\left\{x^2 + R^2 - 2xR\cos\varphi\right\}^{3/2}}$$

$$B_z\Big|_{y=0} = \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} d\varphi \frac{R - x\cos\varphi}{\left\{x^2 + R^2 - 2xR\cos\varphi\right\}^{3/2}}$$

The result is exactly the same as eq. (6) in ref. 1.